

# How to find the effective size of a non-Weyl graph

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# Description of the model

- set of ordinary differential equations
- graph consists of set of vertices  $\mathcal{V}$ , set of not oriented edges (both finite  $\mathcal{E}$  and infinite  $\mathcal{E}_\infty$ ).
- Hilbert space of the problem

$$\mathcal{H} = \bigoplus_{(j,n) \in I_{\mathcal{L}}} L^2([0, l_{jn}]) \oplus \bigoplus_{j \in I_{\mathcal{C}}} L^2([0, \infty)).$$

- states described by columns

$$\psi = (f_{jn} : \mathcal{E}_{jn} \in \mathcal{E}, f_{j\infty} : \mathcal{E}_{j\infty} \in \mathcal{E}_\infty)^T.$$

- the Hamiltonian acting as  $-\frac{d^2}{dx^2} + V(x)$ , where  $V(x)$  is bounded and supported only on the internal edges – corresponds to the Hamiltonian of a quantum particle for the choice  $\hbar = 1$ ,  $m = 1/2$

# Domain of the Hamiltonian

- domain consisting of functions in  $W^{2,2}(\Gamma)$  satisfying coupling conditions at each vertex
- coupling conditions given by

$$(U_v - I_v)\Psi_v + i(U_v + I_v)\Psi'_v = 0.$$

where  $\Psi_v = (\psi_1(0), \dots, \psi_d(0))^T$  and  $\Psi'_v = (\psi_1(0)', \dots, \psi_d(0)')^T$  are the vectors of limits of functional values and outgoing derivatives where  $d$  is the number edges emanating from the vertex  $v$  and  $U_v$  is a unitary  $d \times d$  matrix

- in particular, standard (Kirchhoff) coupling conditions:  
 $f_i(v) = f_j(v), \sum_{j=1}^d f'_j(v) = 0.$

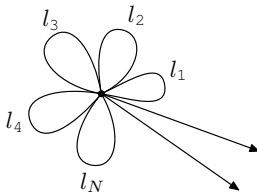
## Flower-like model

- description of coupling conditions using one-vertex graphs
- suppose that  $\Gamma$  has  $N$  internal and  $M$  external edges
- the coupling condition

$$(U - I)\Psi + i(U + I)\Psi' = 0$$

describes coupling on the whole graph;  $U$  is  $(2N + M) \times (2N + M)$  unitary matrix consisting of blocks  $U_v$

- the above equation decouples into conditions for particular vertices
- $U$  encodes not only coupling at the vertices, but also the topology of the graph



## Effective coupling on a finite graph

- replacing non-compact graph by its compact part
- instead of halflines there are effective coupling matrices
- $N$  internal,  $M$  external edges –  $U$  consists of blocks  $U_j$

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

where  $U_1$  is  $2N \times 2N$  matrix corresponding to coupling between internal edges, etc.

- by a standard procedure (Schur, etc.) one gets an effective coupling matrix

$$\tilde{U}(k) = U_1 - (1 - k)U_2[(1 - k)U_4 - (k + 1)I]^{-1}U_3$$

- coupling condition has the same form only with  $U$  replaced by  $\tilde{U}(k)$

$$(\tilde{U}(k) - I)\Psi_1 + i(\tilde{U}(k) + I)\Psi'_1 = 0$$

# Resolvent resonances

- poles of the meromorphic continuation of the resolvent  $(H - \lambda \text{id})^{-1}$
- can be obtained by the external complex scaling – transformation external components of the wavefunction

$$g_j(x) \mapsto U_\theta g_j(x) = e^{\theta/2} g_j(xe^\theta)$$

with a nontrivial imaginary part of  $\theta$

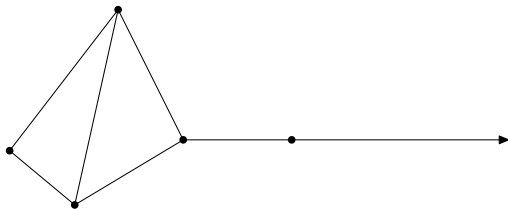
- another definition:  $\lambda = k^2$  is a resolvent resonance if there exists a generalized eigenfunction  $f \in L^2_{\text{loc}}(\Gamma)$ ,  $f \neq 0$  satisfying  $-f''(x) = k^2 f(x)$  on all edges of the graph and fulfilling the coupling conditions, which on all external edges behaves as  $c_j e^{ikx}$ .

# Asymptotics of resonances on quantum graphs

- $N(R)$  number of resolvent resonances in the circle of radius  $R$  in the  $k$  plane
- expected behaviour of the counting function

$$N(R) = \frac{2\text{vol}(\Gamma)}{\pi}R + \mathcal{O}(1)$$

- trivial example of a graph where this asymptotics is not satisfied



# Asymptotics for standard conditions

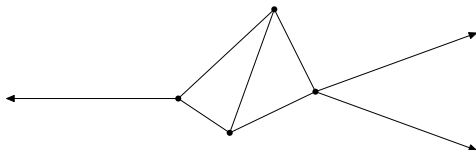
## Theorem (Davies, Pushnitski)

Suppose that  $\Gamma$  is the graph with standard (Kirchhoff) coupling conditions at all the vertices. Then

$$N(R) = \frac{2}{\pi}WR + O(1), \quad \text{as } R \rightarrow \infty,$$

where the coefficient  $W$  satisfies  $0 \leq W \leq \text{vol}(\Gamma)$ . The behaviour is non-Weyl ( $0 \leq W < \text{vol}(\Gamma)$ ) iff there is a balanced vertex.

- $\text{vol}(\Gamma)$  is sum of lengths of the internal edges
- balanced vertex: number of internal edges is equal to the number of external edges





# Pseudo orbit expansion for the resonance condition

- there is a known method for finding the spectrum of a compact graph by the pseudo orbit expansion
- the vertex scattering matrix maps the vector of amplitudes of the incoming waves into a vector of amplitudes of the outgoing waves  $\vec{\alpha}_v^{\text{out}} = \sigma^{(v)} \vec{\alpha}_v^{\text{in}}$
- for a non-compact graph we similarly define effective vertex scattering matrix  $\tilde{\sigma}^{(v)}$

## Theorem

*Let us assume the vertex connecting  $n$  internal and  $m$  external edges. The effective vertex-scattering matrix is given by*

$$\tilde{\sigma}(k) = -[(1-k)\tilde{U}(k) - (1+k)I_n]^{-1}[(1+k)\tilde{U}(k) - (1-k)I_n]$$

*In particular, for the standard conditions we have*

*$\tilde{\sigma}(k) = \frac{2}{n+m}J_n - I_n$ , where  $J_n$  denotes  $n \times n$  matrix with all entries equal to one. For a balanced vertex we have  $\tilde{\sigma}(k) = \frac{1}{n}J_n - I_n$ .*

- idea of the pseudo orbit expansion: replacing the compact part of the graph  $\Gamma$  by a oriented graph  $\Gamma_2$ , each edge replaced by two bonds  $b, \hat{b}$
- ansatz

$$f_{b_j}(x) = \alpha_{b_j}^{\text{in}} e^{-ikx} + \alpha_{b_j}^{\text{out}} e^{ikx},$$

$$f_{\hat{b}_j}(x) = \alpha_{\hat{b}_j}^{\text{in}} e^{-ikx} + \alpha_{\hat{b}_j}^{\text{out}} e^{ikx}$$

- due to the relation  $f_{b_j}(x) = f_{\hat{b}_j}(\ell_j - x)$  we have

$$\alpha_{b_j}^{\text{in}} = e^{ik\ell_j} \alpha_{\hat{b}_j}^{\text{out}}, \quad \alpha_{\hat{b}_j}^{\text{in}} = e^{ik\ell_j} \alpha_{b_j}^{\text{out}}.$$

- we define  $\tilde{\Sigma}(k)$  as a block-diagonalizable matrix written in the basis corresponding to

$$\vec{\alpha} = (\alpha_{b_1}, \dots, \alpha_{b_N}, \alpha_{\hat{b}_1}, \dots, \alpha_{\hat{b}_N})^T$$

which is block diagonal with blocks  $\tilde{\sigma}_v(k)$  if transformed to the basis

$$(\alpha_{b_{v_1 1}}^{\text{in}}, \dots, \alpha_{b_{v_1 d_1}}^{\text{in}}, \alpha_{b_{v_2 1}}^{\text{in}}, \dots, \alpha_{b_{v_2 d_2}}^{\text{in}}, \dots)^T.$$

- we define

$$Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

scattering matrix  $S = Q\tilde{\Sigma}$  and

$$L = \text{diag}(\ell_1, \dots, \ell_N, \ell_1, \dots, \ell_N)$$

- we obtain

$$\begin{pmatrix} \vec{\alpha}_b^{\text{in}} \\ \vec{\alpha}_{\hat{b}}^{\text{in}} \end{pmatrix} = e^{ikL} \begin{pmatrix} \vec{\alpha}_b^{\text{out}} \\ \vec{\alpha}_{\hat{b}}^{\text{out}} \end{pmatrix} = e^{ikL} Q \begin{pmatrix} \vec{\alpha}_b^{\text{out}} \\ \vec{\alpha}_{\hat{b}}^{\text{out}} \end{pmatrix} = e^{ikL} Q \tilde{\Sigma}(k) \begin{pmatrix} \vec{\alpha}_b^{\text{in}} \\ \vec{\alpha}_{\hat{b}}^{\text{in}} \end{pmatrix}$$

- the resonance condition therefore is

$$\det(e^{ikL} Q \tilde{\Sigma}(k) - I_{2N}) = 0$$

# Effective size for the equilateral graph

## Theorem

*Let  $\Gamma$  be an equilateral graph (with all internal edges of length  $\ell$ ). Then the effective size of this graph is  $\frac{\ell}{2} n_{\text{nonzero}}$ , where  $n_{\text{nonzero}}$  is the number of nonzero eigenvalues of the matrix  $Q\tilde{\Sigma}$ . Note that the rank of the matrix cannot be used instead of number of nonzero eigenvalues, because the matrix often has a Jordan form.*

idea of the proof

- if there is  $n_{\text{zero}}$  eigenvalues 0 of  $Q\tilde{\Sigma}$ , then one has to take  $n_{\text{zero}}$  entries from the unit matrix to the determinant

- **periodic orbit**  $\gamma$  is a closed path on  $\Gamma_2$
- **pseudo orbit**  $\tilde{\gamma}$  is a collection of periodic orbits
- **irreducible pseudo orbit**  $\tilde{\gamma}$  is a pseudo orbit, which does not use any directed edge more than once
- we define length of a periodic orbit by  $\ell_\gamma = \sum_{j, b_j \in \gamma} \ell_j$ ; the length of pseudo orbit (and hence irreducible pseudo orbit) is the sum of the lengths of the periodic orbits from which it is composed
- we define product of scattering amplitudes for a periodic orbit  $\gamma = (b_1, b_2, \dots, b_n)$  as  $A_\gamma = S_{b_2 b_1} S_{b_3 b_2} \dots S_{b_1 b_n}$ , where  $S_{b_2 b_1}$  is the entry of the matrix  $S$  in the  $b_2$ -th row and  $b_1$ -th column; for a pseudo orbit we define  $A_{\tilde{\gamma}} = \prod_{\gamma_n \in \tilde{\gamma}} A_{\gamma_j}$
- by  $m_{\tilde{\gamma}}$  we denote the number of periodic orbits in the pseudo orbit  $\tilde{\gamma}$

- reformulation of the theorem on the resonance condition

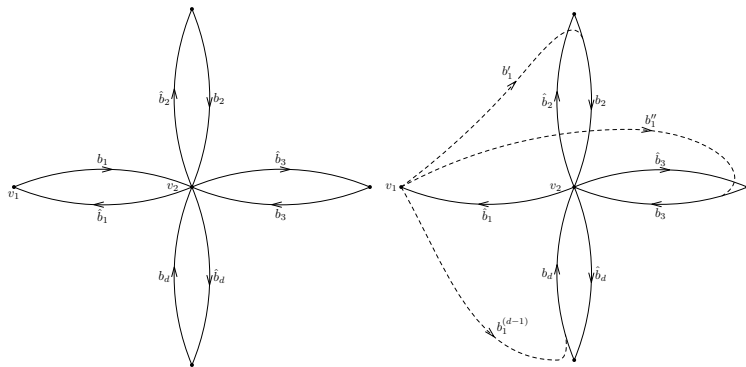
### Theorem

*The resonance condition is given by the sum over irreducible pseudo orbits*

$$\sum_{\bar{\gamma}} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} e^{ik\ell_{\bar{\gamma}}} = 0.$$

- in general  $A_{\bar{\gamma}}$  can be energy dependent, but this is not the case for standard coupling.
- idea of the proof: the permutations in the determinant can be represented as product of disjoint cycles

# Deleting edges of the graph and “ghost edges”



# Main results

## Theorem

*Let us assume the equilateral graph (lengths  $\ell$ ) with standard coupling. Then its effective size is  $W \leq N\ell - \frac{\ell}{2}n_{\text{bal}} - \frac{\ell}{2}n_{\text{nonneigh}}$ , where  $n_{\text{bal}}$  is the number of balanced vertices and  $n_{\text{nonneigh}}$  is the number of balanced vertices which do not neighbour any other balanced vertex.*

- for each balanced vertex one directed edge entering this vertex can be deleted
- in the balanced vertex of the degree  $d$  which do not neighbour any other balanced vertex we have  $d - 1$  incoming directed bonds and  $d$  outgoing directed bonds
- no ghost edge ends in the outgoing edge
- for each balanced vertex one directed edge cannot be used in the irreducible pseudo orbit



## Theorem

*Let us assume the equilateral graph (lengths  $\ell$ ) with standard coupling which contains four balanced vertices which form a square (vertex 1 is connected with 2, 2 with 3, 3 with 4, 4 with 1, but vertex 1 is not connected with 3 and 2 is not connected with 4). Then the effective size is  $W \leq (N - 3)\ell$ .*

- we delete four directed edges of the square
- we use the symmetry of the graph with “ghost edges” and cancellation of contributions of some irreducible pseudo orbits

Thank you for your attention!

## Articles on which the talk was based

E.B. Davies, A. Pushnitski: Non-Weyl Resonance Asymptotics for Quantum Graphs, *Analysis and PDE* **4** (2011), no. 5, pp. 729–756. arXiv: 1003.0051.

E.B. Davies, P. Exner, J. Lipovský: Non-Weyl asymptotics for quantum graphs with general coupling conditions, *J. Phys. A: Math. Theor.* **43** (2010), 474013. arXiv: 1004.0856.

J. Lipovský: On the effective size of a non-Weyl graph, *J. Phys. A: Math. Theor.* **49** (2016), 375202. arXiv: 1507.04176

J. Lipovský: Pseudo orbit expansion for the resonance condition on quantum graphs and the resonance asymptotics, *Acta Physica Polonica A* **128** (2015), p. 968–973. arXiv: 1507.06845

R. Band, J. M. Harrison, and C. H. Joyner: Finite pseudo orbit expansion for spectral quantities of quantum graphs, *J. Phys. A: Math. Theor.* **45** (2012), p. 325204. arXiv: 1205.4214